13.4 Position, Velocity, Acceleration

If ***t* = *time*** and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$$

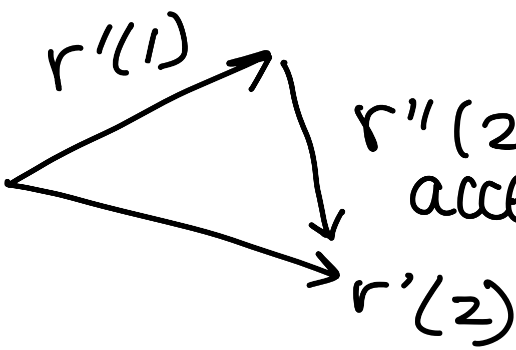
then

$$\boldsymbol{r}'(t) = \lim_{h\to 0}\frac{\overbrace{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}^{\text{displacement}}}{\underbrace{h}_{\text{time}}} = \text{velocity}$$

$$|\boldsymbol{r}'(\boldsymbol{t})| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed} = \frac{\text{ds}}{\text{dt}}$$

$$\boldsymbol{r}''(t) = \lim_{h\to 0}\frac{\boldsymbol{r}'(t+h)-\boldsymbol{r}'(t)}{h}$$

$$= \frac{\text{change in velocity}}{\text{change in time}} = \boldsymbol{a}(t)$$



acceleration!

Entry Task:

Let *t* be **time in seconds** and assume the position of an object (in **feet**) is given by

$$\boldsymbol{r}(t) = < \underset{x}{t}, \underset{y}{2e^{-t}}, \underset{z}{0} >.$$

$r(0) = <0,2,0>$

Compute $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$, and $\boldsymbol{r}''(t)$

and $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$, and $\boldsymbol{r}''(0)$.

$r'(t) = <1, -2e^{-t}, 0>$

$|r'(t)| = \sqrt{1+4e^{-2t}}$

$r''(t) = <0, 2e^{-t}, 0>$

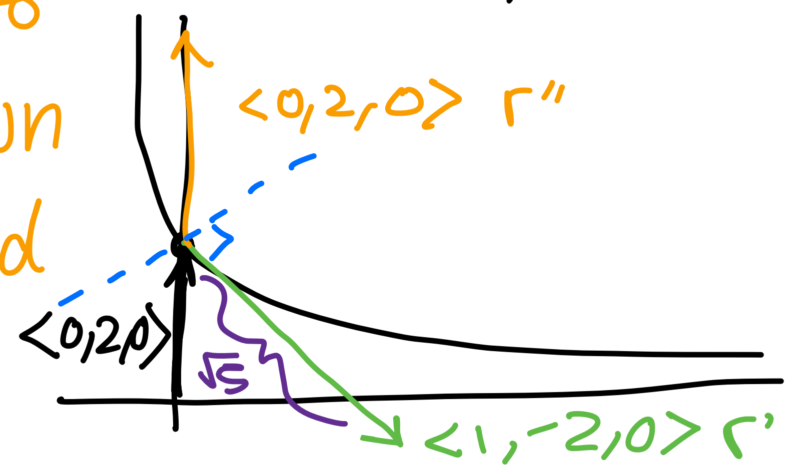
$r'(0) = <1, -2, 0>$

$r'(0) = \sqrt{1+4} = \sqrt{5}$

$r''(0) = <0, 2, 0>$

causing velocity to slow down b/c behind

Normal vector



HUGE application:

ANY motion problem

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \text{ , so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant)

If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve a differential equation) to find velocity and position.

That is how we can model ALL motion problems!

HW Example:

An object of mass 10 kg is being acted on by the force

$$\boldsymbol{F} = \langle 130t, 10e^t, 10e^{-t} \rangle.$$

You are given

$\boldsymbol{v}(0) = \langle 0,0,1 \rangle$ and $\boldsymbol{r}(0) = \langle 0,1,1 \rangle$.

Find the position function.

$$\vec{a}(t) = \frac{1}{10} < 130t, 10e^t, 10e^{-t} >$$

$$\vec{a}(t) = < 13t, e^t, e^{-t} >$$

$$\vec{v}(t) = < \frac{13}{2}t^2 + C_1, e^t + C_2, -e^{-t} + C_3 >$$

$v(0) = 0 + C_1 = 0 \quad C_1 = 0$

$v(0) = 1 + C_2 = 0 \quad C_2 = -1$

$v(0) = -1 + C_3 = 1 \quad C_3 = 2$

$$\vec{v}(t) = < \frac{13}{2}t^2, e^t - 1, -e^{-t} + 2 >$$

$$\boxed{\vec{r}(t) = < \frac{13}{6}t^3, e^t - t, e^{-t} + 2t >}$$

this kind of Q won't be on test

For a much more interesting and applied example try this one…

A ball with mass $m = 0.8$ kg is thrown northward into the air with initial speed of 30 m/sec at an angle of 30 degrees with the ground. A west wind applies a steady force of 4 N on the ball (west to east).

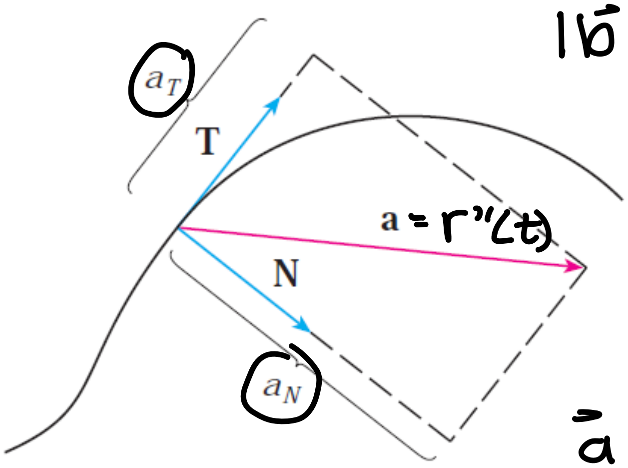
If you are standing on level ground, where does the ball land?

I won't do this in class, but here is a video of me working through it as well as visuals: [Applied Motion Example]

Visuals: https://www.math3d.org/QbuedSnK

Tangential and Normal Components of Acceleration

$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$



$\frac{\vec{a} \cdot \vec{T}}{|\vec{T}|}$

r'' $\quad \frac{r'}{|r|}$

Definition:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$ = tangential comp.

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$ = normal comp.

which can be rewritten as…

$$a_T = \frac{\boldsymbol{r}' \cdot \boldsymbol{r}''}{|\boldsymbol{r}'|} \quad \text{and} \quad a_N = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|}$$

Notes:

- a_T can be positive or negative (or zero)
  positive – speedometer speed increasing
  negative – speedometer speed decreasing

- a_N is always positive (or zero)
  (accel. points "inward" relative to the curve, but not always "directly" inward)

For interpreting use,

$a_T = v' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = kv^2$ = curvature · (speed)^2

(derivative of this is on the next page)

Deriving interpretations
(you can skip this):

Note visually that you can see:

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$$

We are trying to find the numbers in front of **T** and **N.**

Let $\nu(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \frac{\vec{\boldsymbol{v}}(t)}{\nu(t)}$ implies $\vec{\boldsymbol{v}} = \nu\vec{\boldsymbol{T}}$.
2. $\kappa(t) = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{T}}'|}{\nu(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa\nu$.
3. $\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \frac{\vec{\boldsymbol{T}}'}{\kappa\nu}$ implies $\vec{\boldsymbol{T}}' = \kappa\nu\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + \nu\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + k\nu^2\vec{\boldsymbol{N}}.$$

Conclusion:

$a_T = \nu' = \frac{d}{dt}|r'(t)| =$ “deriv. of speed”

$a_N = k\nu^2 =$ curvature · (speed)^2

Example:

$$\vec{r}(t) = < \cos(t), \sin(t), t >$$

Find the tangential and normal components of acceleration.

$$\frac{r' \cdot r''}{|r'|} \qquad \frac{|r' \times r''|}{|r'|}$$

$r'(t) = < -\sin(t), \cos(t), 1 >$

$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$r''(t) = < -\cos(t), -\sin(t), 0 >$

$r' \cdot r'' = \cos(t)\sin(t) - \cos(t)\sin(t) + 0 = 0$

constant speed

$\underbrace{r' \times r''}_{\Downarrow \text{ plug in \#s if given, messy}} = \sqrt{2}$

$\frac{\sqrt{2}}{\sqrt{2}} = 1 = 2K$

$K = 1/2$

F'18 – Exam 2 – Loveless

Consider the position vector

$\boldsymbol{r}(t) = \langle 5t, e^{t}, e^{-3t}\rangle$.

Find all values of at which the tangential component of acceleration is zero.

$$a_T = \frac{r' \cdot r''}{|r'|} = 0$$

$$r' \cdot r'' = 0$$

$$r'(t) = \langle 5, e^{t}, -3e^{-3t} \rangle$$

$$r''(t) = \langle 0, e^{t}, 9e^{-3t} \rangle$$

$$0 + e^{2t} - 27e^{-6t} = 0$$

$$\left(e^{2t} - \frac{27}{e^{6t}} = 0\right)e^{6t}$$

$$e^{8t} = 27$$

$$8t = \ln(27)$$

$$\boxed{t = \frac{1}{8}\ln(27)}$$

Visuals: https://www.math3d.org/iq2UMCKN